Shape Recognition Using Eigenvalues of the Laplacian ♢

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Neuchatel, June 2009
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Lectures

1. Lecture 1: The Dirichlet Laplacian as a Model Problem for Shape Recognition
2. Lecture 2: Numerical Schemes and Statistically Recognizing Shape
3. Lecture 3: Shape Recognition Using Neumann and Higher Order Eigenvalue Problems

What is shape recognition?

- Shape recognition is a key component of (automated) object recognition, matching, and analysis
- A shape description method generates a feature vector that attempts to uniquely characterize the shape of an object
- This is one of the least developed areas of Pattern Recognition

A good feature vector associated with an object should be ..

- invariant under scaling
- invariant under rigid motion (rotation and translation)
- tolerant to noise and reasonable deformation
- should react differently to images from different classes, producing feature vectors different from class to class
- use least number of features to design faster and simpler classification algorithms

Feature Vectors Based on Eigenvalues of Elliptic Operators

- We will build feature vectors out of eigenvalues
- We think of a shape as a domain $\Omega \subset \mathbb{R}^2$
- We think of a shape as a *binary image*
- Four model problems will be presented
- For each, four model features vectors will be studied
- We will illustrate feature recognition schemes for synthetic, and real images and compare results for the various model problems

The Dirichlet Laplacian as a Model Problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $d \geq 2$. Consider the Dirichlet (or Fixed Membrane) Problem:

$$\begin{aligned} -\Delta u &= \lambda \ u \quad in \quad \Omega \\ u &= 0 \quad on \quad \partial\Omega \end{aligned} \tag{1}$$

Eigenmodes: $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$
Eigenfunctions: $u_1, u_2, u_3, \cdots$.
One can characterize these eigenvalues using the *Rayleigh-Ritz Principle*:

$$\lambda_{k+1} \leq \frac{\int_\Omega |\nabla\phi|^2 dx}{\int_\Omega \phi^2 dx}$$

subject to

$$\int_\Omega \phi \, u_i dx = 0, \qquad \phi = 0 \text{ on } \partial\Omega$$

for $i = 1, 2, \ldots, k$.

Some inequalities and stability results:

For $\Omega \subset \mathbb{R}^2$:
Rayleigh-Faber-Krahn Inequality (1890s, 1920's):

$$\lambda_1 \geq \frac{\pi j_{0,1}^2}{|\Omega|}$$

where $j_{0,1} = 2.4048\ldots$

Ashbaugh-Benguria (1991) inequality (formerly PPW conjecture, 1956)

$$\frac{\lambda_2}{\lambda_1} \leq \frac{j_{1,1}^2}{j_{0,1}^2} = 2.53873\ldots.$$

Here $j_{1,1} = 3.83171\ldots$These are isoperimetric inequalities:
Equality holds when Ω is a disk.
A. Melas (1992, 1993) proved stability results for these inequalities when Ω is convex. (These results hold when $\Omega \subset \mathbb{R}^d$.)

The Counting Function and Riesz Means

Theorem (Weyl, 1910/1911)

$$\lambda_k \sim \frac{4\pi^2 k^{2/d}}{(C_d|\Omega|)^{2/d}} = \frac{k^{2/d}}{\left(L_{0,d}^{cl}|\Omega|\right)^{2/d}} \text{ as } k \to \infty,$$

where $C_d = \dfrac{\pi^{d/2}}{\Gamma(d/2+1)}$ = volume of the $d-$Ball.
One can recast this theorem in terms of the counting function:

$$N(z) = \sum_{\lambda_k \leq z} 1 = \sup_{\lambda_k \leq z} k.$$

$$N(z) \sim L_{0,d}^{cl}\,|\Omega|\,z^{d/2} \text{ as } z \to \infty$$

with $L_{0,d}^{cl} = C_d/(2\pi)^d$.

The Riesz mean is a "smoothed staircase" function.

By convention, the counting function is sometimes written as

$$N(z) = \sum_k (z - \lambda_k)_+^0 \, .$$

The reason for this is to parallel the definition of the *Riesz mean of order* $\rho > 0$

$$R_\rho(z) = \sum_k (z - \lambda_k)_+^\rho \, .$$

Here $x_+ = \max\{0, x\}$ is called the *ramp function*.

Properties:

(i)

$$R_\rho(z) = \rho \int_0^\infty (z - t)_+^{\rho - 1} \, N(t) dt.$$

(ii)

$$R_{\sigma+\delta}(z) = \frac{\Gamma(\sigma + \delta + 1)}{\Gamma(\sigma + 1)\ \Gamma(\delta)} \int_0^\infty (z - t)_+^{\delta - 1} \, R_\sigma(t) dt.$$

Riesz Means, cont'd

Remark: (i) These properties are sometimes referred to as *Riesz iteration* or the *Aizenman-Lieb procedure*. These are *Riemann-Liouville fractional transforms* (see the Bateman Project, Vol. I)
(ii) Formulas rely on Fubini and the definition of the Beta function.

Basic references: (1) Article by Dirk Hundertmark in Barry Simon's Festschrift (2006); (2) **"Typical Means"** by Chandrasekharan & Minakshisundaram (1954).

$$R_\rho(z) \sim L^{cl}_{\rho,d} \, |\Omega| \, z^{\rho+d/2} \text{ as } z \to \infty$$

with $L^{cl}_{\rho,d} = \frac{\Gamma(\rho+1)}{(4\pi)^{d/2}\,\Gamma(\rho+d/2+1)}$.

Kac and Berezin-Li-Yau

Heuristic argument: Apply Laplace transform

$$Z(t) = \text{ partition function } = \sum_{k=1}^{\infty} e^{-\lambda_k t} = \int_0^{\infty} e^{-t\mu} N(\mu) d\mu.$$

One then gets Kac's asymptotic formula (see "Can one hear the shape of a drum?", 1966)

$$Z(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \sim \frac{|\Omega|}{(4\pi t)^{d/2}}.$$

$z \to \infty$ corresponds to $t \to 0+$

Theorem (Berezin). For $\rho \geq 1$, one has

$$R_\rho(z) \leq L^{cl}_{\rho,d} \; |\Omega| \; z^{\rho+d/2},$$

Idea of proof: Prove for $\rho = 1$, then apply Riesz iteration.

Berezin-Li-Yau (Laptev-Weild, Journées EDP, 2000)

Let:

$$\hat{u}_k(\xi) = \frac{1}{(2\pi)^d} \int_\Omega u_k(x)\, e^{ix\cdot\xi} dx.$$

Clearly

$$\lambda_k = \int_{\mathbb{R}^d} |\xi|^2 |\hat{u}_k(\xi)|^2 d\xi \quad \text{and} \quad \int_{\mathbb{R}^d} |\hat{u}_k|^2 d\xi = 1$$

Therefore

$$\begin{aligned} \sum_k (z-\lambda_k)_+ &= \sum_k \left(\int_{\mathbb{R}^d} \left(z - |\xi|^2\right) |\hat{u}_k(\xi)|^2 d\xi \right)_+ \\ &\le \int_{\mathbb{R}^d} \left(z - |\xi|^2\right)_+ \sum_k |\hat{u}_k(\xi)|^2 d\xi \end{aligned}$$

where Jensen's inequality is used for every individual integral.
Finish with

$$\sum_k |\hat{u}_k(\xi)|^2 = \frac{1}{(2\pi)^d} \int_\Omega |e^{-ix\cdot\xi}|^2\, dx = \frac{|\Omega|}{(2\pi)^d}.$$

Legendre Transform

Definition: The Legendre transform is defined by:

$$\Lambda\{f\}(w) = \sup_{z \geq 0} \left(w\ z - f(z) \right).$$

Basic properties: (i)

$$f(z) \leq g(z) \Rightarrow \Lambda\{f\}(w) \geq \Lambda\{g\}(w).$$

(ii)

$$\Lambda\Big\{ \sum_i (z - \lambda_i)_+ \Big\}(w) = (w - [w])\ \ \lambda_{[w]+1} + \sum_{i=1}^{[w]} \lambda_i,$$

(iii)

$$\Lambda\{c\, \frac{z^{1+d/2}}{1+d/2}\} = c^{-2/d}\, \frac{d}{d+2}\, w^{1+2/d}$$

Inequalities of Li-Yau and Kac

Applying the *Legendre Transform* to the Berezin inequality (1972) leads to the

Corollary (Li-Yau inequality, 1983):

$$\sum_{i=1}^{k} \lambda_i \geq \frac{d}{d+2} \frac{4\pi^2 k^{1+2/d}}{(C_d|\Omega|)^{2/d}}.$$

Corollary (Kac, 1966):

$$Z(t) \leq \frac{|\Omega|}{(4\pi\, t)^{d/2}}$$

Proof: Apply Laplace transform to Berezin inequality.

Corollary: For $0 < \rho < 1$

$$R_\rho(z) \leq F_{\rho,d}\, L^{cl}_{\rho,d}\ |\Omega|\ z^{\rho+d/2}.$$

Remark: Frank, Loss, Weidl (2008) have the best constant $F_{\rho,d}$.

Some of the Tools Used to Estimate Eigenvalues

Rayleigh-Ritz Ratio: For f defined on Ω such that $f = 0$ on $\partial\Omega$

$$R(f) = \frac{\int_\Omega |\nabla f|^2 dx}{\int_\Omega f^2 dx}$$

Poincaré (1904): For a complete family of functions $g_1, g_2, \ldots, g_n, \ldots$ vanishing along $\partial\Omega$ form

$$\phi = \sum_{j=1}^{n} t_j g_j$$

This leads to

$$R(\phi) = \frac{\sum_{i,j=1}^{n} \; a_{ij} \, t_i t_j}{\sum_{i,j=1}^{n} \; b_{ij} \, t_i t_j}$$

where

$$a_{ij} = \int_\Omega \nabla g_i \cdot \nabla g_j \, dx \qquad b_{ij} = \int_\Omega g_i g_j dx.$$

With $A = (a_{ij})$ and $B = (b_{ij})$, form the equation

$$\left| A - \lambda B \right| = 0$$

Some of the Tools Used to Estimate Eigenvalues

The roots $\lambda'_1 \leq \lambda'_2 \leq \ldots \leq \lambda'_n$ of this equation are such that

$$\lambda_1 \leq \lambda'_1, \quad , \lambda_2 \leq \lambda'_2, \ldots, \lambda_n \leq \lambda'_n$$

Minimax Principle (Fischer, 1905): Formulation preferred by Finite Difference people

$$\lambda_k \leq Min_{S_k}\, max_{\phi \in S_k}\, R(\phi)$$

where S_k is the $k-$dimensional linear space generated by $g_1, g_2, \ldots, g_k$

Maximin Principle (Courant): Formulation preferred by analysts/geometers

$$\lambda_k \leq Max_{T_{k-1}}\, Min_{\phi \perp T_{k-1}} R(\phi)$$

where T_{k-1} is a $k-1$ dimensional linear space and $\phi = 0$ on $\partial\Omega$.

Universal Eigenvalue Bounds

Payne-Pólya-Weinberger (1956)

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{d} \left(\frac{1}{k} \sum_{j=1}^{k} \lambda_j \right) \qquad \text{and} \qquad \frac{\lambda_{k+1}}{\lambda_k} \leq 1 + \frac{4}{d}$$

Hile-Protter (1981)

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{dk}{4}$$

H.C. Yang (1991/1995)

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{d} \sum_{i=1}^{k} \lambda_i (\lambda_{k+1} - \lambda_i)$$

$$\lambda_{k+1} \leq \left(1 + \frac{4}{d}\right) \overline{\lambda}_k$$

Universal Eigenvalue Bounds

Harrell-Stubbe (1997), Ashbaugh-H., For $\rho \geq 2$

$$\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^\rho \leq \frac{2\rho}{d} \sum_{i=1}^{k} \lambda_i(\lambda_{k+1} - \lambda_i)^{\rho-1}$$

For $\rho \leq 2$

$$\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^\rho \leq \frac{4}{d} \sum_{i=1}^{k} \lambda_i(\lambda_{k+1} - \lambda_i)^{\rho-1}$$

Variational Proof: Test function + "Optimal Cauchy-Schwarz":
$\phi_i = xu_i - \sum_{j=1}^{k} \alpha_{ij} u_j$, where $\alpha_{ij} = \langle xu_i, u_j \rangle$, and $x = x_1, \ldots, x_d$ the coordinate functions.
Commutator Proof (a.k.a. sum rules of quantum mechanics):
Technique pioneered by Harrell-Stubbe, followed by
Levitin-Parnovski, El-Soufi-Harrell-Ilias, Harrell-H., Harrell-Yolcu.

Commutators

$$[A, B] = AB - BA$$

First and Second Commutation:

$$[-\Delta, x_\alpha] = -2\frac{\partial}{\partial x_\alpha}$$

$$[[-\Delta, x_\alpha], x_\alpha] = -2$$

Consequence:

$$(\lambda_m - \lambda_j)\,\langle x_\alpha u_j, u_m\rangle = \langle [-\Delta, x_\alpha] u_j, u_m\rangle$$

Commutators, cont'd

Proof (brief):

$$\sum_j (z - \lambda_j)_+^2 \langle [-\Delta, x_\alpha] u_j, x_\alpha u_j \rangle \leq \sum_j (z - \lambda_j)_+ \|[-\Delta, x_\alpha] u_j\|^2$$

Use first commutation formula to get:

$$\|[-\Delta, x_\alpha] u_j\|^2 = 4 \int_\Omega \left(\frac{\partial u_j}{\partial x_\alpha} \right)^2$$

Use second commutation formula to get:

$$\langle [-\Delta, x_\alpha] u_j, x_\alpha u_j \rangle = \int_\Omega u_j^2 = 1$$

Sum over $\alpha = 1, \ldots, d$ to get

- $$\sum_j (z - \lambda_j)_+^2 \leq \frac{4}{d} \sum_j \lambda_j (z - \lambda_j)_+$$

Monotonicity Principle for Riesz Means

- For $\rho \geq 2$ and $z \geq \lambda_1$,

$$\sum_j (z - \lambda_j)_+^\rho \leq \frac{2\rho}{d} \sum_j \lambda_j (z - \lambda_j)_+^{\rho-1}$$

  and consequently

$$\frac{R_\rho(z)}{z^{\rho + \frac{d}{2}}}$$

  is a nondecreasing function of z.
- For $\rho \leq 2$ and $z \geq \lambda_1$,

$$\sum_j (z - \lambda_j)_+^\rho \leq \frac{4}{d} \sum_j \lambda_j (z - \lambda_j)_+^{\rho-1}$$

  and consequently

$$\frac{R_\rho(z)}{z^{\rho + \frac{\rho d}{4}}}$$

  is a nondecreasing function of z.

Sum Rules vs Rayleigh-Ritz

- One can get these from first principles through sum rules (Harrell-Stubbe, Levitin-Parnovski, Harrell-H., El-Soufi-Harrell-Ilias, Harrell-Stubbe, extensions by Harrell-Yolcu);
- Alternative way via Rayleigh-Ritz: Ashbaugh-H., Colbois, Ilias-Makhoul, Cheng-Yang, Cheng-Yang-Sun, Wang-Xu, Wu, Wu-Cao, Jöst-Li-Jöst-Wang-Xu, etc.
- Sum rules + Integral transforms: One can obtain all from the $\rho = 2$ case (for the model problem)
- These are particular cases of more general monotonicity principles for "trace controllable functions" as shown in recent work by Harrell-Stubbe

What does the monotonicity principle entail?

It leads universal bounds for ratios of eigenvalues which are of Weyl-type.

- (Harrell-H., 2008) For $k \geq j \geq 1$,

$$\lambda_{k+1}/\overline{\lambda_j} \leq \left(1 + \frac{4}{d}\right) \left(\frac{k}{j}\right)^{\frac{2}{d}}.$$

case $j = 1$ (Cheng-Yang, 2007); case $j = k$ (Yang, 91/95)

- (Harrell-H., 2008) For $k \geq j \frac{1+\frac{d}{2}}{1+\frac{d}{4}}$,

$$\overline{\lambda_k}/\overline{\lambda_j} \leq 2 \left(\frac{1 + \frac{d}{4}}{1 + \frac{d}{2}}\right)^{1+\frac{2}{d}} \left(\frac{k}{j}\right)^{\frac{2}{d}}.$$

- Harrell-Stubbe (2009): For $k \geq j$,

$$\overline{\lambda_k}/\overline{\lambda_j} \leq \frac{1 + \frac{d}{4}}{1 + \frac{d}{2}} \left(\frac{k}{j}\right)^{\frac{2}{d}}.$$

Proof of λ_{k+1} bound

Let n be the largest such that $\lambda_n \leq z < \lambda_{n+1}$, then

$$R_2(z) = n\left(z^2 - 2z\overline{\lambda_n} + \overline{\lambda_n^2}\right).$$

For any integer j and $z \geq \lambda_j$,

$$R_2(z) \geq Q(z,j) := j\left(z^2 - 2z\overline{\lambda_j} + \overline{\lambda_j^2}\right).$$

By monotonicity, for $z \geq z_j \geq \lambda_j$,

$$R_2(z) \geq Q(z_j,j)\left(\frac{z}{z_j}\right)^{2+\frac{d}{2}}.$$

Also, by Cauchy-Schwarz $\overline{\lambda_j}^2 \leq \overline{\lambda_j^2}$, so

$$Q(z,j) = j\left(\left(z - \overline{\lambda_j}\right)^2 + \overline{\lambda_j^2} - \overline{\lambda_j}^2\right)$$

$$\geq j\left(z - \overline{\lambda_j}\right)^2.$$

Proof of the λ_{k+1} bound, cont'd

Combining and choosing $z = z_j = \left(1 + \frac{4}{d}\right) \overline{\lambda_j}$, one gets

$$R_2(z) \geq \frac{j z^{2+\frac{d}{2}}}{\left(1 + \frac{d}{4}\right)^2 \left(\left(1 + \frac{4}{d}\right) \overline{\lambda_j}\right)^{\frac{d}{2}}}.$$

From monotonicity, one gets

$$R_1(z) \geq \left(1 + \frac{d}{4}\right) \frac{1}{z} R_2(z),$$

and,

$$N(z) = R_0(z) \geq \left(1 + \frac{d}{4}\right)^2 \frac{1}{z^2} R_2(z)$$

and therefore,

$$N(z) \geq j \left(\frac{z}{\left(1 + \frac{4}{d}\right) \overline{\lambda_j}}\right)^{\frac{d}{2}}.$$

To get the bound statement for λ_{k+1}, simply send $z \to \lambda_{k+1}$ from below.

Three Basic Messages

1. (Integral) transforms link various inequalities proved by various techniques

$$\begin{array}{ccc} \text{Yang} & \Leftrightarrow & \text{Harrell-Stubbe, } \rho \geq 2 \\ \Downarrow & & \Downarrow \\ \text{Kac} & \Leftrightarrow & \text{Berezin-Li-Yau, } \rho \geq 2 \end{array}$$

They provide a parallel framework to convexity.

2. Sum rules play a key role.

3. By Legendre transform, any bound for a Riesz mean of order $\rho = 1$ which is of Weyl-type can be converted to statements about ratios of eigenvalues (or ratios of means of eigenvalues) which are of Weyl-type.

Riesz iteration: $\rho = 2$ **implies** $\rho > 2$**:**

$$\sum_k (z - \lambda_k)_+^2 \leq \frac{4}{d} \sum_k \lambda_k \, (z - \lambda_k)_+ \, ,$$

Therefore, for $t \leq z$:

$$\sum_k (z - \lambda_k - t)_+^2 \leq \frac{4}{d} \sum_k \lambda_k \, (z - \lambda_k - t)_+ \, .$$

Multiply both sides by $t^{\rho-3}$, and then integrate between 0 and ∞.

$$\sum_k (z - \lambda_k)_+^\rho \leq \frac{4}{d} \, \frac{\Gamma(\rho+1)\Gamma(2)}{\Gamma(\rho)\Gamma(3)} \sum_k \lambda_k \, (z - \lambda_k)_+^{\rho-1} \, .$$

With $\Gamma(\rho + 1) = \rho \, \Gamma(\rho)$, this simplifies to

$$\sum_k (z - \lambda_k)_+^\rho \leq \frac{2\rho}{d} \, \sum_k \lambda_k \, (z - \lambda_k)_+^{\rho-1} \, ,$$

Note: The constant in this inequality is the sharpest possible.

$\rho = 2$ implies $\rho < 2$:

This is a consequence of the "Weighted Reverse Chebyshev Inequality":
Let $\{a_k\}$ and $\{b_k\}$ be two real sequences, one of which is nondecreasing and the other nonincreasing, and let $\{w_k\}$ be a sequence of nonnegative weights. Then,

$$\sum_{k=1}^{m} w_k \sum_{k=1}^{m} w_k \; a_k b_k \leq \sum_{k=1}^{m} w_k \; a_k \sum_{k=1}^{m} w_k \; b_k .$$

Make the choices $w_k = (z - \lambda_k)_+^{\rho_1}$, $a_k = \frac{\lambda_k}{(z-\lambda_k)_+}$, and $b_k = (z - \lambda_k)_+^{\rho_2 - \rho_1}$ with $\rho_1 \leq \rho_2 \leq 2$, the conditions of the lemma are satisfied and one gets:

$$\frac{\sum_k (z - \lambda_k)_+^{\rho_1}}{\sum_k (z - \lambda_k)_+^{\rho_1 - 1} \;\; \lambda_k} \leq \frac{\sum_k (z - \lambda_k)_+^{\rho_2}}{\sum_k (z - \lambda_k)_+^{\rho_2 - 1} \;\; \lambda_k} .$$

then, set $\rho_1 = \rho$ and $\rho_2 = 2$

Basic message, revisited

Berezin-Li-Yau (for $\rho \geq 2$) follows from Harrell-Stubbe, and semiclassical asymptotic formula.

- For $\rho \geq 2$ and $z \geq \lambda_1$,

$$\frac{R_\rho(z)}{z^{\rho+\frac{d}{2}}}$$

  is a nondecreasing function of z.

- $$\lim_{z\to\infty} \frac{R_\rho(z)}{z^{\rho+\frac{d}{2}}} = L^{cl}_{\rho,d}\,|\Omega|$$

Harrell-Stubbe + Asymptotic ⇒ Kac's inequality

Apply the Laplace transform to both sides of

$$\sum_{k=1}^{\infty} (z-\lambda_k)_+^2 \leq \frac{4}{d} \sum_{k=1}^{\infty} \lambda_k \, (z-\lambda_k)_+ ,$$

and use

$$\mathcal{L}\left((z-\lambda_k)_+^{\rho}\right) = \frac{\Gamma(\rho+1)\; e^{-\lambda_k \; t}}{t^{\rho+1}}.$$

to obtain

$$Z(t) \leq -\frac{2}{d}\, t\, Z'(t)$$

or, after combining,

$$\left(t^{d/2}\, Z(t)\right)' \leq 0.$$

then employ

$$\lim_{t\to 0+} \; t^{d/2} Z(t) = \frac{|\Omega|}{(4\pi)^{d/2}}.$$

Harrell-Stubbe + Asymptotic $\Rightarrow$ Kac's inequality

Therefore $t^{d/2}Z(t)$ is a nonincreasing function which saturates when $t \to 0$:

$$Z(t) \leq \frac{|\Omega|}{(4\pi t)^{d/2}}$$

This is Kac's inequality.

From Berezin-Li-Yau to Kac's

Start with

$$R_\rho(\lambda) \le L^{cl}_{\rho,d}\, |\Omega|\, \lambda^{\rho+d/2}$$

Apply the Laplace transform to both sides

$$\frac{\Gamma(\rho+1)}{t^{\rho+1}} Z(t) \le L^{cl}_{\rho,d}\, |\Omega| \frac{\Gamma(\rho+1+\frac{d}{2})}{t^{\rho+1+\frac{d}{2}}}.$$

Upon simplification, it obtains

$$Z(t) \le \frac{|\Omega|}{t^{\frac{d}{2}}}\, \frac{L^{cl}_{\rho,d}\, \Gamma(\rho+1+\frac{d}{2})}{\Gamma(\rho+1)}.$$

Using the definition of $L^{cl}_{\rho,d}$ leads to Kac's inequality.

Monotonicity + Kac's Asymptotic ⇒ Berezin-Li-Yau, when $\rho \geq 2$:

$$R_\rho(\mu + z_0) \geq R_\rho(z_0) \left(\frac{\mu + z_0}{z_0}\right)^{\rho + d/2}.$$

The *Laplace transform* of a shifted function

$$\mathcal{L}\left(f(\mu + z_0)\right) = e^{z_0 t}\left(\mathcal{L}(f) - \int_0^{z_0} e^{-t\mu} f(\mu) d\mu\right)$$

Therefore, for each individual term on the LHS, we obtain

$$\begin{aligned}\mathcal{L}\left((\mu + z_0 - \lambda_k)_+^\rho\right) &= e^{(z_0 - \lambda_k)_+ t}\left(\frac{\Gamma(\rho+1)}{t^{\rho+1}}\right. \\ &- \left.\int_0^{(z_0 - \lambda_k)_+} e^{-t\mu} \mu^\rho d\mu\right).\end{aligned}$$

Monotonicity + Kac's Asymptotic $\Rightarrow$ Berezin-Li-Yau, when $\rho \geq 2$:

On the RHS, one has

$$\mathcal{L}\left((\mu + z_0)^{\rho+d/2}\right) = e^{z_0\,t}\Big(\frac{\Gamma(\rho+1+d/2)}{t^{\rho+1+d/2}} - \int_0^{z_0} e^{-t\mu}\mu^{\rho+d/2}d\mu\Big).$$

We note the appearance of the incomplete γ function

$$\gamma(a,x) = \int_0^x e^{-\mu}\mu^{a-1}d\mu.$$

Putting these facts together we are led to

$$\sum_k e^{(z_0-\lambda_k)_+ t}\left\{\frac{\Gamma(\sigma+1)}{t^{\sigma+1}} - \frac{1}{t^{\rho+1}}\gamma\left(\sigma+1,(z_0-\lambda_k)_+ t\right)\right\} \geq$$

$$\frac{R_\sigma(z_0)}{z_0^{\rho+d/2}}e^{z_0\,t}\left\{\frac{\Gamma(\rho+1+d/2)}{t^{\rho+1+d/2}} - \frac{1}{t^{\rho+1+d/2}}\gamma(\rho+1+d/2, z_0\,t)\right\}.$$

Monotonicity + Kac's Asymptotic ⇒ Berezin-Li-Yau, when $\rho \geq 2$:

We now notice that

$$\sum_k e^{(z_0-\lambda_k)_+ t} \leq e^{z_0\, t} \sum_{k=1}^{\infty} e^{-\lambda_k t} = e^{z_0\, t}\, Z(t).$$

Therefore, after a little simplification,

$$\frac{\Gamma(\sigma+1)}{\Gamma(\rho+1+d/2)}\, t^{d/2} Z(t) \geq \frac{R_\sigma(z_0)}{z_0^{\rho+d/2}} + \mathcal{R}(t),$$

where the remainder term $\mathcal{R}(t)$ is given by the long expression

$$\begin{aligned} \mathcal{R}(t) &= \frac{t^{d/2}}{\Gamma(\rho+1+d/2)} e^{-z_0 t} \sum_k e^{(z_0-\lambda_k)_+ t}\, \gamma(\sigma+1, (z_0-\lambda_k)_+\, t) \\ &- \frac{t^{d/2}}{\Gamma(\rho+1+d/2)} \frac{R_\sigma(z_0)}{z_0^{\rho+d/2}}\, \gamma(\rho+1+d/2, z_0\, t) \end{aligned}$$

Notice that $\lim_{t\to 0} \mathcal{R}(t) = 0$. Sending $t \to 0$, and incorporating Kac's semiclassical leads to result.

Integral Transforms and Universal Lower Bounds for Riesz Means

Remember some of the spectral functions we dealt with

- The counting function $N(z)$
- The Riesz Mean of order ρ: Riemann-Louiville fractional transform of $N(z)$
- The "partition function" $Z(t)$
- The spectral zeta function

$$\zeta_{spec}(\rho) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{\rho}}$$

This is the Mellin transform of the $Z(t)$.

A General Setting for New Universal Inequalities

For a nonnegative function f on $\mathbb{R}_+$ such that

$$\int_0^\infty f(t)\left(1+t^{-d/2}\right)\frac{dt}{t} < \infty$$

define

$$F(s) := \int_0^\infty e^{-st} f(t) \frac{dt}{t} \tag{2}$$

and let

$$G(s) := \mathcal{W}_{d/2}\{F(z)\}(s), \tag{3}$$

where

$$\mathcal{W}_\mu\{F(z)\}(s) := \frac{1}{\Gamma(\mu)} \int_s^\infty F(z)\,(z-s)^{\mu-1}\,dz$$

denotes the *Weyl transform of order* μ of the function $F(z)$.
Bateman project:

$$G(s) = \int_0^\infty \frac{e^{-st}}{t^{d/2}} f(t) \frac{dt}{t}.$$

Universal Lower Bounds amenable to the above setting:

Theorem (Harrell-H.): For $\rho \geq 1$

$$R_\rho(z) \geq H_d^{-1} \frac{\Gamma(1+\rho)\Gamma(1+d/2)}{\Gamma(1+\rho+d/2)} \lambda_1^{-d/2} (z-\lambda_1)_+^{\rho+d/2} .$$

Here:

$$H_d = \frac{2\ d}{j_{d/2-1,1}^2 J_{d/2}^2(j_{d/2-1,1})}.$$

As usual, $j_{\alpha,p}$ denotes the p-th positive zero of the Bessel function $J_\alpha(x)$.

$$Z(t) \geq \frac{\Gamma(1+d/2)}{H_d} \frac{e^{-\lambda_1 t}}{(\lambda_1\ t)^{d/2}}.$$

For $\rho > d/2$

$$\zeta_{spec}(\rho) \geq \frac{\Gamma(1+d/2)}{H_d} \frac{\Gamma(\rho-d/2)}{\Gamma(\rho)} \frac{1}{\lambda_1^\rho}.$$

This provides correction for the zeta function when ρ is close to $d/2$.

Universal Lower Bounds Via Weyl transforms

For $F(s)$ and $G(s)$ as defined above, and related by the Weyl transform,

$$\sum_{j=1}^{\infty} F(\lambda_j) \geq \frac{\Gamma(1+d/2)}{H_d} \lambda_1^{-d/2} G(\lambda_1).$$

Note: This inequality is equivalent to the partition function bound found above.

Work in Progress: The Neumann Case

For $\rho \geq 1$

$$\sum_{j=1}^{\infty} (z - \mu_j)_+^{\rho} \geq L_{\rho,d}^{cl} \, |\Omega| \, z^{\rho + d/2}.$$

$$\sum_{j=1}^{\infty} e^{-\mu_j t} \geq \frac{|\Omega|}{(4\pi t)^{d/2}}.$$

For $\rho > d/2$,

$$\zeta_{Hur}(\rho) = \sum_{j=1}^{\infty} \frac{1}{(\mu_j + \alpha)^{\rho}} \geq \frac{\Gamma(\rho - d/2)}{(4\pi)^{d/2}\Gamma(\rho)} \, \frac{|\Omega|}{\alpha^{\rho - d/2}}.$$

For $F(s)$ and $G(s)$ as defined above, and related by the Weyl transform, and $\alpha > 0$

$$\sum_{j=1}^{\infty} F(\mu_j + \alpha) \geq \frac{|\Omega|}{(4\pi)^{d/2}} \; G(\alpha).$$

From Bethe Sum Rule to a Theorem of Laptev:

Our starting point is the Bethe sum rule (see for example, Levitin-Parnovski, 2002)

$$\sum_k (\lambda_k - \lambda_j) \, | \int_\Omega u_k u_j e^{ix\cdot\xi} dx|^2 = |\xi|^2.$$

This provides alternative proof of the following result of Laptev (There are other proofs by L. H., '08, Frank-Laptev-Molchanov, '09)

Theorem [Laptev, 96]

$$\sum_j (z - \lambda_j)_+ \geq L_{1,d}^{cl} \, \tilde{u}_1^{-2} \, (z - \lambda_1)_+^{1+d/2} \, . \tag{4}$$

where $\tilde{u}_1 = \text{ess sup}|u_1|$ and $L_{1,d}^{cl}$ is the classical constant.

From Bethe Sum Rule to Universal Inequalities:

Proof: Let

$$a_{jk}(\xi) = \int_\Omega u_k u_j e^{ix\cdot\xi} dx$$

Take $j = 1$.

$$\sum_k (\lambda_k - \lambda_1) |a_{1k}(\xi)|^2 = |\xi|^2.$$

Let $z > \lambda_1$. One can always find an integer N such that

$$\lambda_N < z \leq \lambda_{N+1},$$

allowing the sum to be split as

$$\sum_k = \sum_{k=1}^{N} + \sum_{k=N+1}^{\infty} .$$

We can replace each term in $\sum_{k=N+1}^{\infty}(\dots)$ by

$$(z - \lambda_1)\ |a_{1k}(\xi)|^2.$$

From Bethe Sum Rule to Universal Inequalities:

Hence

$$\sum_{k=1}^{N} (\lambda_k - \lambda_1) \ |a_{1k}(\xi)|^2 + (z - \lambda_1) \ \left(1 - \sum_{k=1}^{N} |a_{1k}(\xi)|^2\right) \leq |\xi|^2.$$

Here we have exploited the completeness of the orthonormal family $\{u_k\}_{k=1}^{\infty}$, noting that

$$\sum_{k=1}^{\infty} |a_{1k}(\xi)|^2 = \int_{\Omega} |u_1 e^{ix\cdot\xi}|^2 = 1.$$

Therefore

$$\sum_{k=N+1}^{\infty} |a_{1k}(\xi)|^2 = 1 - \sum_{k=1}^{N} |a_{1k}(\xi)|^2.$$

These identities reduce our inequality to

$$(z - \lambda_1)_+ \leq |\xi|^2 + \sum_{k} (z - \lambda_k)_+ \, |a_{1k}(\xi)|^2. \qquad (5)$$

(The statement is true by default for $z \leq \lambda_1$.)

From Bethe Sum Rule to Universal Inequalities:

One then integrates over a ball $B_r \subset \mathbb{R}^d$ of radius r. To simplify the notation we use

$$|B_r| = \text{volume of } B_r = C_d \, r^d,$$

and

$$I_2(B_r) = \int_{B_r} |\xi|^2 d\xi = \frac{d}{d+2} C_d \, r^{d+2}.$$

Our main inequality then reduces to

$$(z - \lambda_1)_+ \leq \frac{I_2(B_r)}{|B_r|} + \sum_k (z - \lambda_k)_+ \frac{\int_{B_r} |a_{1k}(\xi)|^2 d\xi}{|B_r|}.$$

By the Plancherel-Parseval identity

$$\begin{aligned}
\frac{1}{(2\pi)^d} \int_{B_r} |a_{1k}(\xi)|^2 d\xi &\leq \int_\Omega |u_1|^2 |u_k|^2 dx \\
&\leq \text{ess sup} |u_1|^2 \int_\Omega |u_k(x)|^2 dx \\
&= \text{ess sup} |u_1|^2.
\end{aligned}$$

From Bethe Sum Rule to Universal Inequalities:

Riesz iteration leads to the corollary:
For $\rho \geq 1$

$$\sum_k (z - \lambda_k)_+^\rho \geq L_{\rho,d}^{cl} \, \tilde{u}_1^{-2} \, (z - \lambda_1)_+^{\rho + d/2} \,. \tag{6}$$

We also have the following *universal lower bound (H., Trans. AMS, 2008)*

$$\sum_k (z - \lambda_k)_+ \geq \frac{2}{d+2} H_d^{-1} \lambda_1^{-d/2} \, (z - \lambda_1)_+^{1+d/2} \,.$$

where

$$H_d = \frac{2 \, d}{j_{d/2-1,1}^2 J_{d/2}^2(j_{d/2-1,1})} . \tag{7}$$

This is a consequence of the Chiti inequality (satisfies Queen Dido property):

$$\tilde{u}_1^2 \leq H_d L_{0,d}^{cl} {\lambda_1}^{d/2}.$$

Work of Melas and corrections to Berezin-Li-Yau

A. Melas (Proc. AMS, 2003) proved the following inequality.

$$\sum_{i=1}^{k} \lambda_i \geq \frac{d}{d+2} \frac{4\pi^2 k^{1+2/d}}{(C_d|\Omega|)^{2/d}} + M_d \frac{|\Omega|}{I(\Omega)} k.$$

Here $I(\Omega)$ is the "second moment" of Ω, while M_d is a constant that depends on the dimension d. This is a correction to BLY. If one applies the Legendre transform to this inequality:

$$R_\rho(z) \leq L^{cl}_{\rho,d} |\Omega| \left(z - M_d \frac{|\Omega|}{I(\Omega)} \right)_+^{\rho + \frac{d}{2}},$$

for $\rho \geq 1$.

The Work of Melas

Applying the Laplace transform leads to the following correction of Kac's inequality

$$\sum_{i=1}^{\infty} e^{-\lambda_i t} \leq \frac{|\Omega|}{(4\pi t)^{d/2}} \, e^{-M_d \frac{|\Omega|}{I(\Omega)} t}. \tag{8}$$

Finally, applying the Mellin transform to this inequality leads to the following

$$\zeta_{spec}(\rho) \leq \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(\rho - d/2)}{\Gamma(\rho)} |\Omega| \left(M_d \frac{|\Omega|}{I(\Omega)} \right)^{\frac{d}{2}-\rho}.$$

In fact we have the general inequality, as above:
For $F(s)$ and $G(s)$ as related by the Weyl transform, one has

$$\sum_{j=1}^{\infty} F(\lambda_j) \leq \frac{1}{(4\pi)^{d/2}} |\Omega| \, G\left(M_d \frac{|\Omega|}{I(\Omega)} \right).$$

Conjectures (For $d \leq 23$ see L. Geisinger and T. Weidl)

$$\sum_{j=1}^{\infty} F(\lambda_j) \leq \frac{1}{(4\pi)^{d/2}} \, |\Omega| \, G(|\Omega|^{-2/d})$$

Here $\frac{1}{|\Omega|^{2/d}}$ replaces $M_d \, \frac{|\Omega|}{I(\Omega)}$. For instance:

1. For $\rho > d/2$,

$$\zeta_{spec}(\rho) \leq \frac{\Gamma(\rho - d/2)}{\Gamma(\rho)} \, \frac{|\Omega|^{2\rho/d}}{(4\pi)^{d/2}}.$$

2. Conjecture(s) would follow from a correction to Kac's inequality:

$$\sum_{i=1}^{\infty} e^{-\lambda_i t} \leq \frac{|\Omega|}{(4\pi t)^{d/2}} \, e^{-\dfrac{t}{|\Omega|^{2/d}}}.$$

3. These would follow from the $\rho \geq 1$ improvement for Riesz means:

$$R_\rho(z) \leq L^{cl}_{\rho,d} |\Omega| \left(z - \frac{1}{|\Omega|^{2/d}} \right)_+^{\rho + \frac{d}{2}}.$$

Conjectures

Iteration on dimension for a parallelpiped

$$\Omega = I_1 \times I_2 \times \cdots \times I_d :$$

$I_1 = [0, \pi]$, $L = \pi$; $L_{1,1}^{cl} = 2/(3\pi)$, $\lambda_k = k^2$.

$$\sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \geq \frac{n^3}{3} + \frac{n}{6}$$

Apply Legendre transform:

$$\sum (z - \lambda_k)_+ \leq \frac{2}{3} \left(z - \frac{1}{6} \right)^{3/2} < L_{1,1}^{cl} \, \pi \, \left(z - \frac{1}{\pi^2} \right)^{3/2}$$

Apply Legendre, etc.
"Lifting" works for $\Omega = \Omega_1 \times \Omega_2$, etc.

$$\lambda_{k\ell} = \mu_k + \nu_\ell.$$

Conjectures

Do they violate any of the known inequalities? No.
Tested against Faber-Krahn, Li-Yau, Pólya (when the domain tiles $\mathbb{R}^d$)

$$\frac{\zeta(2\rho/d)}{(4\pi^2)^\rho}\, C_d^{2\rho/d} \leq \frac{1}{(4\pi)^d}\, \frac{\Gamma(\rho - d/2)}{\Gamma(\rho)} \leq \left(\frac{d+2}{d}\right)^\rho\, \frac{\zeta(2\rho/d)}{(4\pi^2)^\rho}\, C_d^{2\rho/d}.$$

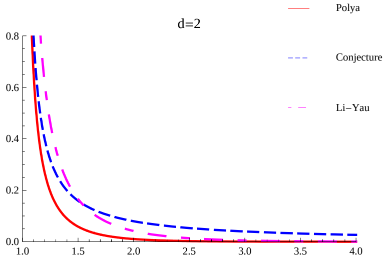

Figure: Upper Bound Estimate for $|\Omega|^{-2\rho/d}\, \zeta_{spec}(\rho)$

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Next Lecture:

- Shape Recognition Using Eigenvalues of the Dirichlet Laplacian
- Finite Difference Schemes for Computing Eigenvalues

Merci!