Lecture 2: Recognizing Shapes Using the Dirichlet Laplacian $\diamondsuit$

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$\diamondsuit$ based on joint work with M. A. Khabou and M. B. H. Rhouma

Summary of Today's Session:

- Properties of Feature Vectors
- Established Techniques for Shape Recognition
- Properties of the Dirichlet Laplacian
- Three Numerical Finite Difference Models for the Dirichlet Problem
- Other Methods
- Algorithm
- Results
- Using the Spectrum to Recognize Shape: Negative and Positive Answers
- Minimax Principle and the Numerical Schemes

Properties of Feature Vectors

Recall ..

A good feature vector associated with an object should be ..

- invariant under scaling
- invariant under rigid motion (rotation and translation)
- tolerant to noise and reasonable deformation
- should react differently to images from different classes, producing feature vectors different from class to class
- use least number of features to design faster and simpler classification algorithms

Established Techniques for Shape Recognition

- boundary methods vs global methods
- Shape measures or descriptors: circularity, rectangularity, ellipticity, triangularity, etc.
- Topological tools
- moments
- Fourier descriptors/wavelet decomposition
- graph theoretical approach

Dirichlet Eigenvalue Problem

Key properties

- Eigenvalues are invariant under rigid motion (translation, rotation)
- Domain monotonicity: If $\Omega_1 \subset \Omega_2$, then $\lambda_k(\Omega_1) \geq \lambda_k(\Omega_2)$.
- For $\alpha > 0$, $\lambda_k(\alpha\,\Omega) = \dfrac{\lambda_k(\Omega)}{\alpha^2}$
- Scale Invariance: $\frac{\lambda_k(\alpha\Omega)}{\lambda_m(\alpha\Omega)} = \frac{\lambda_k(\Omega)}{\lambda_m(\Omega)}$
- All sorts of universal constraints on the eigenvalues

Feature Vectors

For a *binary image* assuming the shape of Ω, consider extracting 4 sets of features. Note that n counts the number of features.

$$F_1(\Omega) = \left(\frac{\lambda_1}{\lambda_2}, \frac{\lambda_1}{\lambda_3}, \frac{\lambda_1}{\lambda_4}, \ldots, \frac{\lambda_1}{\lambda_n} \right)$$

$$F_2(\Omega) = \left(\frac{\lambda_1}{\lambda_2}, \frac{\lambda_2}{\lambda_3}, \frac{\lambda_3}{\lambda_4}, \ldots, \frac{\lambda_{n-1}}{\lambda_n} \right)$$

$$F_3(\Omega) = \left(\frac{\lambda_1}{\lambda_2} - \frac{d_1}{d_2}, \frac{\lambda_1}{\lambda_3} - \frac{d_1}{d_3}, \frac{\lambda_1}{\lambda_4} - \frac{d_1}{d_4}, \ldots, \frac{\lambda_1}{\lambda_n} - \frac{d_1}{d_n} \right)$$

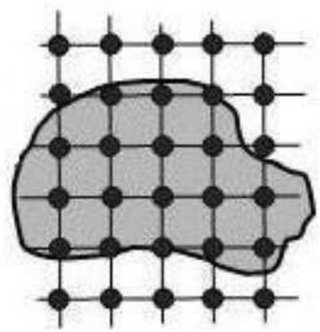
Here $d_1 \leq d_2, \ldots \leq d_n$ are the first n e-values of a disk.

$$F_4(\Omega) = \left(\frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{2\lambda_1}, \frac{\lambda_4}{3\lambda_1}, \ldots, \frac{\lambda_{n+1}}{n\lambda_1} \right)$$

(F_4 scales down the Weyl growth of the eigenvalues.)

Three Numerical Finite Difference Models for the Dirichlet Problem

Let $h > 0$. Pixelize the plane into lattice points (ih, jh), with i, j integers. Let Ω_h is a square grid covering Ω,
$\partial\Omega_h$: pixels through which $\partial\Omega$ passes,and N_h is the number of pixels that cover Ω.



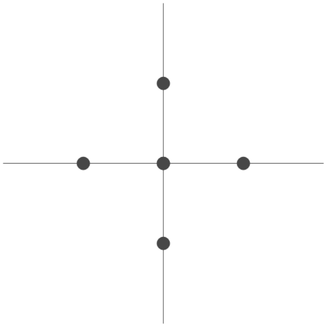
Finite Difference Schemes

5-Point Finite Difference Approximation:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Replace Δ with 5-point finite difference approximation Δ_h defined by:

$$\Delta_h u := \frac{u(x+h) + u(x-h, y) + u(x, y-h) + u(x, y+h) - 4u(x, y)}{h^2}$$



Finite Difference Models for the Dirichlet Problem, cont'd

With u_{ij} denoting the value of u at a lattice point (ih, jh), the discretization takes the form:

$$(\Delta_h u)_{i,j} = \frac{1}{h^2}(u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{ij})$$

Symbolically, we write it in the form:

$$\Delta_h = \frac{1}{h^2} \begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix}$$

The eigenvalue problem is replaced by a matrix eigenvalue problem

$$\begin{aligned} -\Delta_h U &= \lambda^h \ U \quad in \quad \Omega_h \\ U &= 0 \quad on \quad \partial\Omega_h \end{aligned} \tag{1}$$

Eigenmodes: $0 < \lambda_1^h < \lambda_2^h \leq \lambda_3^h \leq \cdots \leq \lambda_{N_h}^h$

What we know: $\Delta - \Delta_h = O(h^2)$

Finite Difference Models for the Dirichlet Problem, cont'd

G. E. Forsythe (1953/4): There exists $\gamma_1, \gamma_2, \ldots, \gamma_k, \ldots$, etc, such that

$$\lambda_k^h \leq \lambda_k - \gamma_k h^2 + o(h^2)$$

the γ_k's cannot be computed, but are positive when Ω is convex.

In fact, we have the following (H. B. Keller, '65):

Theorem: If $\tau_h(\phi(P)) := (\Delta - \Delta_h)\phi(P)$ denotes the *local truncation error*, for a given function ϕ, and point $P \in \Omega_h$, then for each λ_k eigenvalue of the continuous problem, there exists λ^h, eigenvalue of the difference problem, such that

$$|\lambda_k - \lambda^h| \leq \frac{\|\tau(u_k)\|_2}{\|u_k\|_2}$$

Finite Difference Schemes: First Modification, cont'd

Modification 1: Pólya (1952): Generalized eigenvalue problem.
One can think of the discretized problem as:

$$\mathcal{L}_{ij}\ u = \lambda\ \mathcal{R}_{ij}\ u.$$

with

$$\mathcal{L}_{ij}\ u = \frac{1}{h^2}\ \left(u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4\ u_{ij}\right),$$

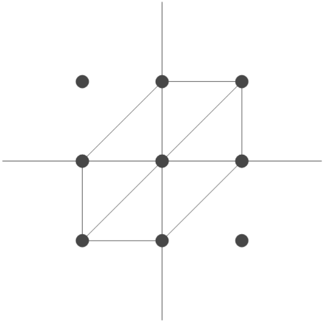
and $\mathcal{R}_{ij} =$ identity. Pólya proposed to change $\mathcal{R}_{ij}$ to:

$$\mathcal{R}_{ij}\ u = -\frac{1}{12}\ \left(6u_{ij} + u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i-1,j-1} + u_{i,j-1}\right).$$

Finite Difference Models for the Dirichlet Problem, cont'd

This takes the form:

$$
\begin{aligned}
-\frac{1}{h^2}\begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix} U &= \frac{\overline{\lambda}^h}{12}\begin{pmatrix} & 1 & 1 \\ 1 & 6 & 1 \\ 1 & 1 & \end{pmatrix} U \text{ in } \Omega_h \\
U &= 0 \text{ on } \partial\Omega_h
\end{aligned}
$$



Finite Difference Models for the Dirichlet Problem, cont'd

Theorem (Pólya, Weinberger): $\lambda_k \leq \overline{\lambda}_k^h \leq \dfrac{\lambda_k}{1 - \frac{1}{4}h^2\lambda_k}$

Corollary:

(1) $\dfrac{\overline{\lambda}_k^h}{1 + \frac{1}{4}h^2\overline{\lambda}_k^h} \leq \lambda_k \leq \overline{\lambda}_k^h$

(2) $\overline{\lambda}_k^h - \lambda_k = O(h^2)$

Theorem (Lyashenko, Embegenov): $\frac{\lambda_k^h + \overline{\lambda}_k^h}{2} = \lambda_k + O(h^4)$ for Ω strictly convex with C^1 boundary.

Finite Difference Models for the Dirichlet Problem, cont'd

Modification 2: Pólya (1952) proposed to replaced both $\mathcal{L}_{ij}$ and $\mathcal{R}_{ij}$ with:

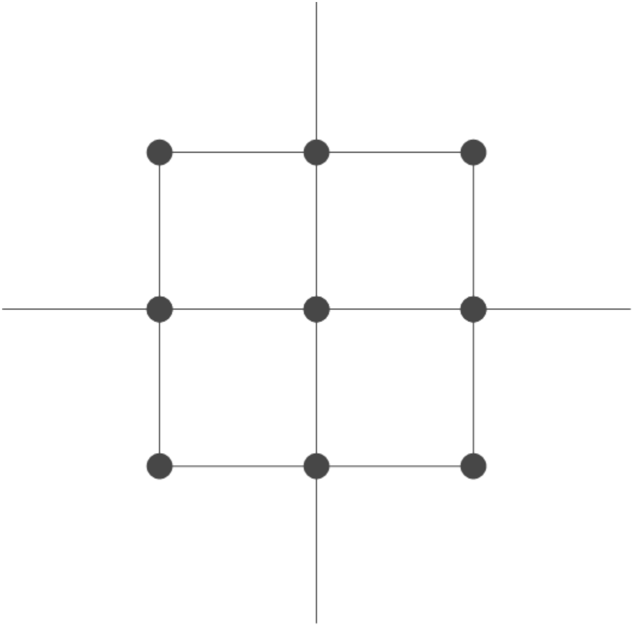
$$\mathcal{L}_{ij}\ u = \frac{1}{3h^2}\left(u_{i+1,j} + u_{i+1,j+1} + u_{i,j+1} + \ldots + u_{i+1,j-1} - 8u_{ij}\right)$$

and

$$\begin{aligned} \mathcal{R}_{ij}\ u = & - \frac{1}{36}\,(16u_{ij} + 4u_{i+1,j} + 4u_{i,j+1} + 4u_{i-1,j} \\ & + 4u_{i,j-1} + u_{i+1,j+1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i-1,j-1}). \end{aligned}$$

$$\begin{aligned} -\frac{1}{3h^2}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -8 & 1 \\ 1 & 1 & 1 \end{pmatrix} U &= \frac{\overline{\lambda}^h}{36}\begin{pmatrix} 1 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 1 \end{pmatrix} U \text{ in } \Omega_h \\ U &= 0 \text{ on } \partial\Omega_h \end{aligned}$$

Finite Difference Schemes: Second Modification, cont'd



Computation for a square of side π

		5×5 mesh			10×10 mesh			20×20 mesh		
			M1	M2		M1	M2		M1	M2
λ_1	2	1.95	2.15	2.05	1.99	2.04	2.01	2.00	2.01	2.00
λ_2	5	4.62	5.80	5.40	4.89	5.24	5.12	4.97	5.07	5.03
λ_3	5	4.62	5.81	5.40	4.89	5.24	5.12	4.97	5.07	5.03
λ_4	8	7.30	10.33	8.75	7.78	8.69	8.22	7.94	8.19	8.06
λ_5	10	8.27	12.84	11.97	9.46	10.85	10.57	9.85	10.23	10.15
λ_6	10	8.27	12.84	11.97	9.46	10.86	10.57	9.85	10.24	10.15
λ_7	13	10.94	18.76	15.32	12.36	14.73	13.67	12.82	13.48	13.18
λ_8	13	10.94	18.76	15.32	12.36	14.75	13.67	12.82	13.48	13.18
λ_9	17	11.92	23.96	21.89	15.33	19.30	18.81	16.53	17.64	17.48

Other Methods of Computation

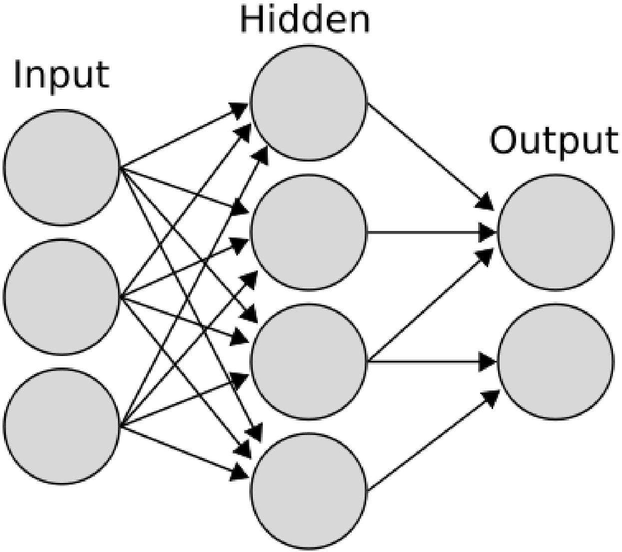
- Finite Elements (Courant, Strang, Strang-Fix, Babuska-Osborn, etc.)
- Method of Particular Solutions (MPS) of Henrici, Fox, Moler (revived by Betcke and Trefethen, '05, Guidotti & Lambers, '08, Saito & Zhang, '09)
- T. Driscoll used a modification of the MPS (of a modification by Descloux & Tolley) to compute the eigenvalues of the isospectral domains (Bilby and Hawk) of Gordon-Webb-Wolpert
- Wu, Sprung, Martorell (1995) used Finite Difference to compute the first 25 evalues of Bilby and Hawk
- Cureton and Kuttler (1999): Conformal transformation techniques (for polygonal domains).

Algorithm and Results

Neural Networks

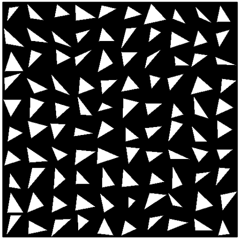
- This is a reliable engineering tool used to classify/label data.
- The process consists of a training/learning phase and a validation/retrieval phase.
- Typically, one divides, randomly, a data set into two subsets: One is used for training and the other one is used for validation.
- A neural network is composed of layers, the number of which depends on the complexity of the data set.

Neural Network

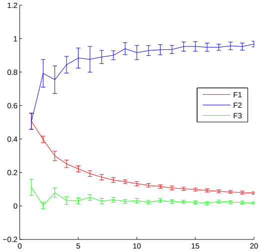


Simple Shape Experiments

- We generated 100 binary images from five classes: disks, ellipses, rectangles, triangles and squares, in random sizes and orientations.
- Some images were so small that it is hard even for a human eye to distinguish them apart
- Computed 20-dimensional vectors for F_1, F_2, and F_3



(a)



(b)

Figure: (a) images of 100 random triangles and (b) the average and standard deviation of the first 20 features from F_1, F_2 and F_3.

Simple Shapes

Table: Correct classification rates of simple shapes using different number of features from F_1, F_2, and F_3 sets.

n	F_1 Features	F_2 Features	F_3 Features
4	96.0%	96.8%	96.0%
8	99.2%	98.4%	97.6%
12	95.2%	95.2%	96.8%
16	97.6%	97.2%	98.4%
20	97.6%	99.2%	98.4%

Tolerance to Noise

- Gauge variation in the boundary of an input image
- Randomly corrup 20 percent of the boundary pixels by either adding or deleting pixels at these locations
- These are more pronounced for small images

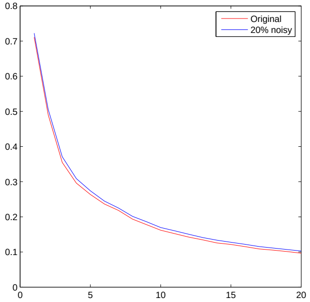


Figure: Noise effects for F_1 features for rectangles

Hand-Drawn Shapes

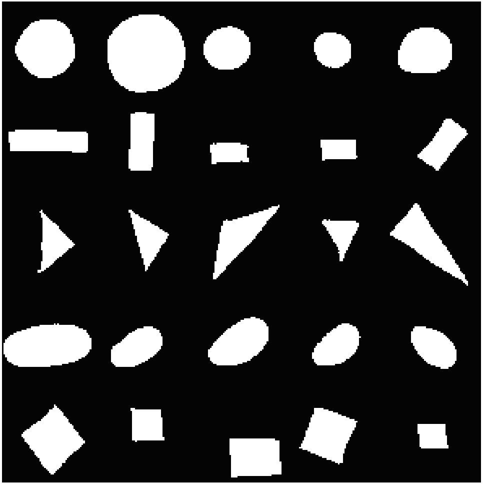


Figure: Samples of the hand-drawn shapes

Hand-Drawn Shapes

Table: Classification results of the hand-drawn shapes.

	F_1 Features	F_2 Features	F_3 Features
Number of features used	12	12	8
Correct classification rate	94.5%	93.5%	94.0%

Synthetic Images: $n-$Petal Shapes

- Defined by: $r = a + \epsilon \cos\theta + \cos n\theta$. Here a measures the size of the interior of the images (randomly chosen between 1 and 2); $0 \leq \epsilon \leq 1$ (randomly chosen), and n is the number of petals.
- We generated five sets of 100 n-petal images for $n = 3, 4, 5, 6,$ and 7 (total of 500 images).

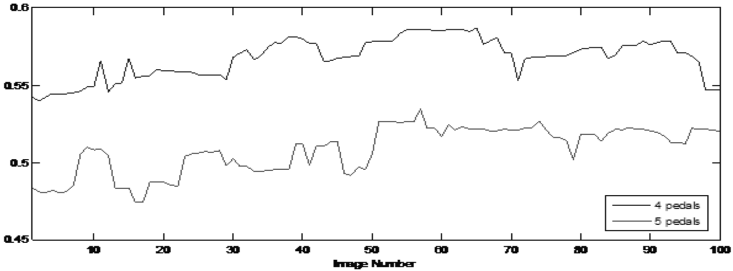


Figure: Plot of the first F_1 feature for all 4-petal and 5-petal images.

$n-$Petal Shapes

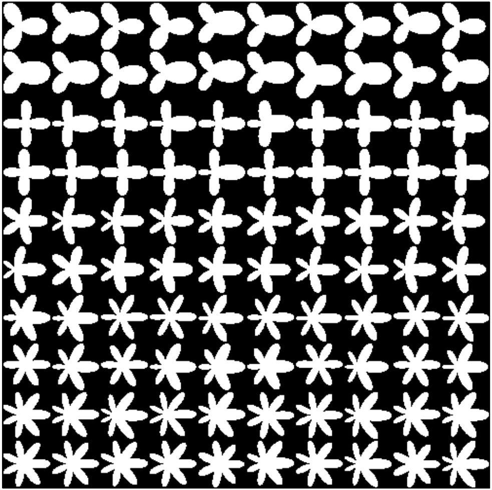


Figure: Sample n-petal images ($n = 3, \ldots, 7$).

$n-$Petal Shapes

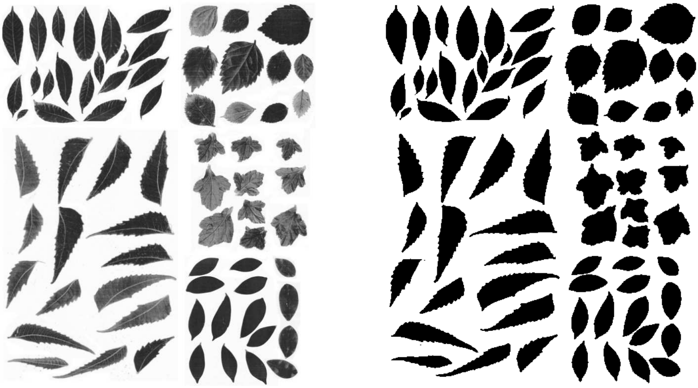
Table: Classification results of the n-petal images ($n = 3, \dots, 7$).

Number of features	F_1 Features	F_2 Features	F_3 Features
4	70.5%	65%	74.5%
8	79.5%	83%	88.5%
12	93%	90%	92%
16	95%	89%	92%
20	97.5%	88%	94.5%

Real Data: Leaf Images

- We have images of leaves from 5 different types of trees, photographed and scanned.
- These images are transformed from gray-scale to binary images (the process is called threshholding) and are then fed into the neural network

Leaf Images



(a) (b)

Figure: Picture of the leaves from 5 different types of trees: (a) gray-scale; (b) threshholded.

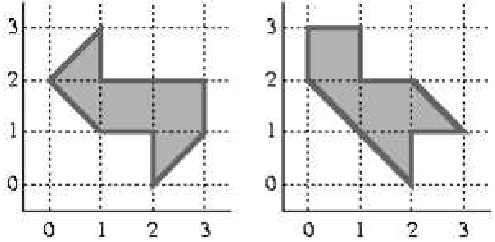
Classification rates for leaf images

Table: Classification results of leaf images.

	F_1 Features	F_2 Features	F_3 Features
Number of features used	2	4	2
Correct classification rate	88.9%	84.7%	88.9%

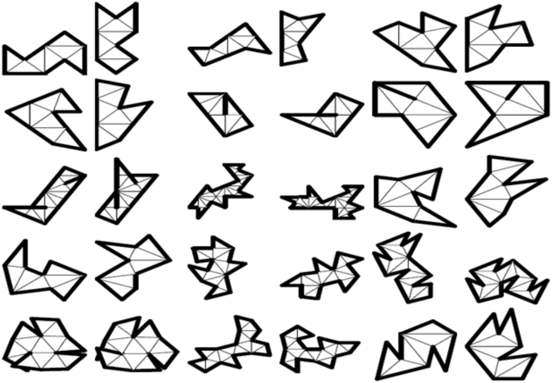
Recognizing Shape: Negative and Positive Answers

- J. Milnor constructed a pair of 16-dimensional tori that have the same eigenvalues but different shapes (1964)
- Bilby and Hawk: Gordon, Webb, and Wolpert (1992): These are a pair of regions in the plane that have different shapes but identical eigenvalues (for the membrane problem); T. Driscoll (1997), and more recently Betcke and Trefethen (2005), checked isospectrality using through computation.



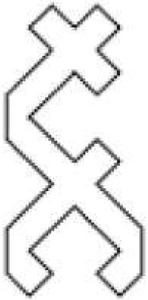
Recognizing Shape: Negative and Positive Answers

- Buser, Conway, Doyle (1994) constructed numerical examples of isospectral 2-d domains.



Recognizing Shape: Negative and Positive Answers

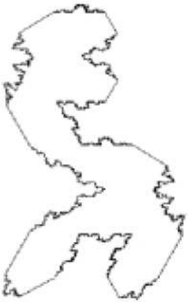
- P. Bérard: Transplantation et isospectralité I, II (1992, 1993)



- H. Urakawa: Bounded domains which are isospectral, but not congruent (early 80s)
- Driscoll-Gottlieb: Isospectral shapes with Neumann and alternating boundary conditions (2003)

Recognizing Shape: Negative and Positive Answers

- Sleeman-Hua: Nonisometric isospectral connected fractal domains (1998, 2000)



Negative and Positive Answers

- S. Zelditch (GAFA, 2000), announcement in Math. Research Letters (99): Under generic conditions, for a family of bounded, simply connected, real analytic plane domains with 4-fold symmetry, the spectrum uniquely determines the underlying domain (up to rigid motion)
- H. Hezari and S. Zelditch (2009): Extension to higher dimensions: "Inverse spectral problem for analytic $(\mathbb{Z}/2\mathbb{Z})^n$ symmetric domains in $\mathbb{R}^n$"

Minimax Principle

$$\lambda_k \leq Min_{g_1,g_2,\ldots,g_k}\, max_{a_1,a_2,\ldots,a_k} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}$$

where $u = a_1 g_1 + a_2 g_2 + \ldots + a_k g_k$

$$\lambda_k^h \leq Min_{g_1,g_2,\ldots,g_k}\, max_{a_1,a_2,\ldots,a_k} \frac{D(v,v)}{h^2 \sum v^2(i,j)}$$

where

$$D(v,v) = \sum_{\Omega_h} (v_{i+1,j} - v_{i,j})^2 + (v_{i,j+1} - v_{i,j})^2$$

and $v = a_1 g_1 + a_2 g_2 + \ldots + a_k g_k$ with $g_1, g_2, \ldots, g_k$ linearly independent mesh functions which vanish off Ω_h. Also $v_{i,j} = v(ih, jh)$.

Finite Difference Models for the Dirichlet Problem, cont'd

Proof of Modification 1 (Idea goes back to L. Collatz '38, Courant, '43, Pólya, '52, Weinberger, '57, Hubbard, '60, Kuttler, '70): Start with mesh *eigenfunctions* $V_1, \ldots, V_k$ of the finite difference problem. Define functions $v_1, \ldots, v_k$ admissible in the continous problem (in the minimax principle). Take each pixel and divide it into two triangles by means of a diagonal in a fixed direction. Let $v_i(x, y)$ be linear on each triangle such that it agrees with the values of the eigenvector V_i at the mesh points. Here

$$v = a_1 v_1 + a_2 v_2 + \ldots + a_k v_k$$

$$V = a_1 V_1 + a_2 V_2 + \ldots + a_k V_k$$

Finite Difference Models for the Dirichlet Problem, cont'd

This lead Pólya to

$$\int_\Omega |\nabla v|^2 dx \le D(V,V)$$

$$\begin{aligned}\int_\Omega v^2 dx &\ge h^2 \sum_{\Omega_h} V_{i,j}^2 - \frac{h^2}{12} \sum_{\Omega_h} \Big\{ (V_{i+1,j} - V_{i,j})^2 \\ &+ (V_{i,j+1} - V_{i,j})^2 + (V_{i+1,j+1} - V_{i,j})^2 \Big\} \qquad (2) \\ &\ge h^2 \sum_{\Omega_h} V_{i,j}^2 - \frac{h^2}{4} D(V,V)\end{aligned}$$

Put these in the minimax principle

$$\begin{aligned}\lambda_k &\le \max_{a_1,a_2,\ldots,a_k} \frac{D(V,V)}{h^2 \sum_{\Omega_h} V_{i,j}^2 - \frac{h^2}{4} D(V,V)} \\ &= \max_{a_1,a_2,\ldots,a_k} \frac{\sum_{i=1}^k a_k^2 \lambda_i^h}{1 - \frac{h^2}{4} \sum_{i=1}^k a_k^2 \lambda_i^h} \le \frac{\lambda_k^h}{1 - \frac{h^2}{4} \lambda_k^h}\end{aligned}$$

Finite Difference Models for the Dirichlet Problem, cont'd

Proof of Modification 2 (Idea goes back to Pólya, '52, details supplied in the book of Forsythe & Wasow, pp. 331-334): For every square mesh with corners U_P, U_E, U_{NE}, and U_N, one constructs a bilinear interpolation, then extend to all of Ω_h

$$u(x,y) = \frac{1}{h^2}\left(U_P(h-x)(h-y) + U_E x(h-y) + U_{NE}xy + U_N(h-x)y\right).$$