Lecture 3: Image Recognition Using Neumann and Higher Order Eigenvalue Problems$\diamondsuit$

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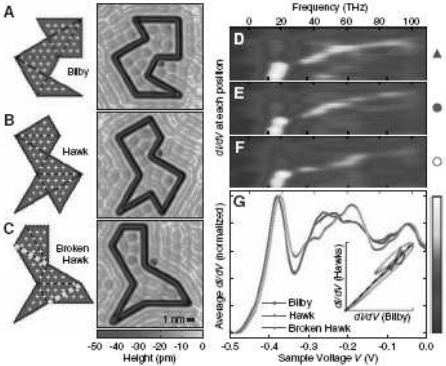
$\diamondsuit$ based on joint work with M. A. Khabou and M. B. H. Rhouma

Summary of Today's Session:

- The Three Other Model Problems
- A flavor of known inequalities
- Universal inequalities (for buckling and clamped plate problems)
- Numerical Schemes
- Feature functions
- Results
- A Unified Approach to Universal Eigenvalues for Second and Higher Order Elliptic Operators

Quantum Drums: Bibly, Hawk, .. and a Broken Hawk

".. Construct quantum isospectral nanostructures with matching electronic structure"



Quantum Phase Extraction in Isospectral Electronic Nanostructures
Christopher R. Moon, *et al.*
Science **319**, 782 (2008);
DOI: 10.1126/science.1151490

They also have a construction for another isospectral pair: Aye-Aye and Beluga

The other model problems

Beyond the Dirichlet eigenvalue problem... one can use:

1. The Free Membrane Problem:

$$-\Delta v = \mu \ v \quad in \quad \Omega$$
$$\frac{\partial v}{\partial n} = 0 \quad on \quad \partial\Omega \tag{1}$$

Eigenmodes: $0 = \mu_1 < \mu_2 \leq \mu_3 \leq \cdots$

The Other Model Problems

2. The Clamped Plate Problem:

$$\Delta^2 w = \Gamma \ w \quad in \quad \Omega \tag{2}$$
$$w = \frac{\partial w}{\partial n} = 0 \quad on \quad \partial\Omega$$

Eigenmodes: $0 < \Gamma_1 \leq \Gamma_2 \leq \Gamma_3 \leq \cdots$

3. The Buckling (of a Clamped Plate) Problem:

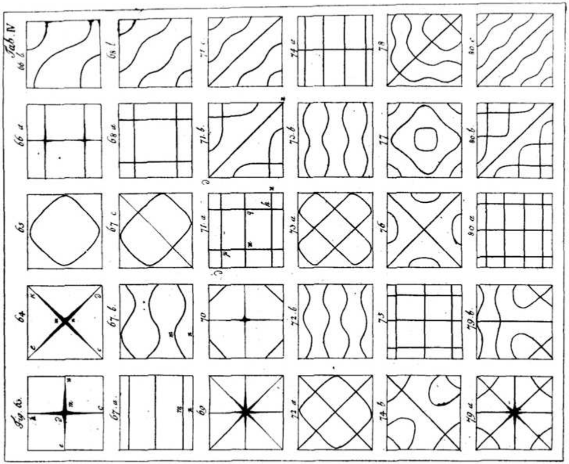
$$\Delta^2 w = -\Lambda \ \Delta w \quad in \quad \Omega \tag{3}$$
$$w = \frac{\partial w}{\partial n} = 0 \quad on \quad \partial\Omega$$

Eigenmodes: $0 < \Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \cdots$

Motivation for Bilaplacian: Chladni Plates

Ernest Chladni of Saxony, “father of accoustics”

His experiments: vibrated a fixed plate with a violin bow and then sprinkled sand across it to show the formation of the nodal lines, mid-1800s (see Bruno Lévy, INRIA)



Rayleigh Quotients

- Clamped Problem

$$R(\phi) = \frac{\int_\Omega (\Delta\phi)^2}{\int_\Omega \phi^2}$$

Apply Cauchy-Schwarz:

$$\left(\int_\Omega |\nabla\phi|^2\right)^2 = \left(-\int_\Omega \phi\Delta\phi\right)^2 \leq \left(\int_\Omega \phi^2\right)\left(\int_\Omega (\Delta\phi)^2\right)$$

So

$$\left(\frac{\int_\Omega |\nabla\phi|^2}{\int_\Omega \phi^2}\right)^2 \leq \frac{\int_\Omega (\Delta\phi)^2}{\int_\Omega \phi^2}$$

Or

$$\lambda_k^2(\Omega) \leq \Gamma_k(\Omega).$$

Weinstein: $\lambda_1^2 \leq \Gamma_1$

Rayleigh Quotients

- Buckling Problem

$$R(\phi) = \frac{\int_\Omega (\Delta\phi)^2}{\int_\Omega |\nabla\phi|^2}$$

Apply Cauchy-Schwarz:

$$\left(\int_\Omega |\nabla\phi|^2\right)^2 = \left(-\int_\Omega \phi\Delta\phi\right)^2 \leq \left(\int_\Omega \phi^2\right)\left(\int_\Omega (\Delta\phi)^2\right)$$

So

$$\frac{\int_\Omega |\nabla\phi|^2}{\int_\Omega \phi^2} \leq \frac{\int_\Omega (\Delta\phi)^2}{\int_\Omega |\nabla\phi|^2}$$

Or

$$\lambda_k(\Omega) \leq \Lambda_k(\Omega)$$

Note: $\lambda_2 \leq \Lambda_1$ (Payne)
See: M. Ashbaugh, "On Universal Inequalities for the Low Eigenvalues of the Bucklng Problem", Partial differential equations and inverse problems, 2004

A flavor of inequalites: Clamped Plate

For simplicity $\Omega \subset \mathbb{R}^2$

- Nadirashvili proved Rayleigh's conjecture

$$\Gamma_1 \geq \frac{\pi^2 k_0^2}{|\Omega|^2}$$

(isoperimetric) with $k_0 = 3.19622062$

- Weyl asymptotic

$$\Gamma_k \approx \frac{16\pi^2 k^2}{|\Omega|^2}$$

- Levine-Protter proved Li-Yau-type inequality (1985)

$$\Gamma_k \geq \frac{16\pi^2 k^2}{3|\Omega|^2}.$$

- Payne-Pólya-Weinberger (1956):

$$\Gamma_{k+1} - \Gamma_k \leq \frac{8}{k} \sum_{j=1}^{k} \Gamma_j, \text{ also } \frac{\Gamma_2}{\Gamma_1} \leq 9$$

A flavor of inequalites: Clamped Plate

- Ashbaugh inequality (1998): $\Gamma_{k+1} - \Gamma_k \leq \frac{8}{k^2}\left(\sum_{j=1}^{k}\sqrt{\Gamma_j}\right)^2$
- Hook and Chen & Qian (1990)

$$\frac{k^2}{8} \leq \left(\sum_{j=1}^{k} \frac{\sqrt{\Gamma_j}}{\Gamma_{k+1} - \Gamma_j}\right) \left(\sum_{j=1}^{k} \sqrt{\Gamma_j}\right)$$

  improves earlier results by Hile-Yeh (1984)
- Cheng-Yang (2006)

$$\Gamma_{k+1} - \frac{1}{k}\sum_{j=1}^{k}\Gamma_j \leq \sqrt{8}\,\frac{1}{k}\sum_{j=1}^{k}\sqrt{\Gamma_j\,(\Gamma_{k+1} - \Gamma_j)}$$

- Wang-Xia (2007)

$$\sum_{j=1}^{k}(\Gamma_{k+1} - \Gamma_j)^2 \leq \sqrt{8}\left(\sum_{j=1}^{k}(\Gamma_{k+1} - \Gamma_j)^2\,\sqrt{\Gamma_j}\right)^{1/2} \times$$

$$\left(\sum^{k}(\Gamma_{k+1} - \Gamma_j)\,\sqrt{\Gamma_j}\right)^{1/2}$$

A flavor of inequalities: Buckling Problem

- Pólya-Szegö Conjecture: $\Lambda_1(\Omega) \geq \Lambda_1(\Omega^\star)$
- Bramble-Payne: $\Lambda_1(\Omega) \geq \frac{2\pi j_{0,1}^2}{|\Omega|}$
- PPW: $\dfrac{\Lambda_2}{\Lambda_1} \leq 3$
- Hile-Yeh: $\dfrac{\Lambda_2}{\Lambda_1} \leq 2.5$
- Ashbaugh: $\dfrac{\Lambda_2 + \Lambda_3}{\Lambda_1} \leq 6$
- Cheng and Yang (2006) proved

$$\sum_{j=1}^{k} \left(\Lambda_{k+1} - \Lambda_j\right)^2 \leq 4 \sum_{j=1}^{k} \Lambda_j \left(\Lambda_{k+1} - \Lambda_j\right).$$

- There is reason to believe that one can improve this inequality to

$$\sum_{j=1}^{k} \left(\Lambda_{k+1} - \Lambda_j\right)^2 \leq 2 \sum_{j=1}^{k} \Lambda_j \left(\Lambda_{k+1} - \Lambda_j\right).$$

Finite Difference Schemes: Neumann Eigenvalues

Remember that we represent all these discretizations in the form:

$$\mathcal{L}_{ij}\ v = \mu^h\ \mathcal{R}_{ij}\ v.$$

$\mathcal{L}$ is called the stiffness matrix, while $\mathcal{R}$ is called the mass matrix. In the Neumann case, $\mathcal{L}$ is still represented by

$$\Delta_h = \frac{1}{h^2}\begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix}$$

and (without modifications) $\mathcal{R}$ is represented by the identity. The normal boundary condition is given (for boundary pixels) by

$$v_{i,j} = \text{ average of adjacent ``interior'' points}$$

e.g.,

$$v_{i,j+1} + v_{i+1,j} + v_{i,j-1} = 3v_{i,j}.$$

Hubbard (1968) carried most of the analysis for the Neumann finite difference scheme (à la Weinberger).

Finite Difference Schemes: Clamped Plate

$$\Delta_h^2 v = \Gamma^h v \tag{4}$$

The result of applying $\Delta_h^2 v = \Delta_h (\Delta_h v)$ is a 13-point discrete scheme:

$$\begin{aligned} h^4\, \Delta_h^2 v &= v(x, y-2h) \\ &+ 2v(x-h, y-h) - 8v(x, y-h) + 2v(x+h, y-h) \\ &+ v(x-2h, y) - 8v(x-h, y) + 20v(x, y) \\ &- 8v(x+h, y) + v(x+2h, y) + 2v(x-h, y+h) \\ &- 8v(x, y+h) + 2v(x+h, y+h) + v(x, y+2h) \end{aligned} \tag{5}$$

Finite Difference Schemes: Clamped Plate, Cont'd

The recursion is given by:

$$\begin{aligned} h^4 \left(\Delta_h^2 v\right)_{ij} &= v_{i,j-2} \\ &+ 2v_{i-1,j-1} - 8v_{i,j-1} + 2v_{i+1,j-1} \\ &+ v_{i-2,j} - 8v_{i-1,j} + 20v_{i,j} \\ &- 8v_{i+1,j} + v_{i+2,j} + 2v_{i-1,j+1} \\ &- 8v_{i,j+1} + 2v_{i+1,j+1} + v_{i,j+2} \end{aligned} \tag{6}$$

The boundary pixels are subject to:

$$v_{i,j} = 0$$

and

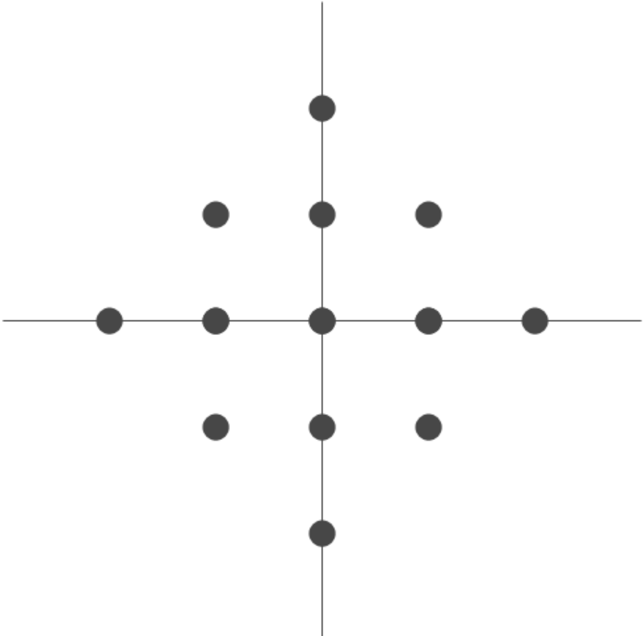
$$v_{i,j} = \text{ average of adjacent ``interior'' points}$$

Finite Difference Schemes: Clamped Plate

The stiffness matrix is represented by:

$$\Delta_h^2 = \frac{1}{h^4} \left(\begin{array}{ccccc} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & 20 & -8 & 1 \\ & 2 & -8 & 2 & \\ & & 1 & & \end{array} \right)$$

Finite Difference Schemes: Clamped Plate



Comparison of known and computed ratios of clamped plate eigenvalues for disks

Known values taken from A. Weinstein (1969)

Γ_1/Γ_2	0.23083388	0.2332	1%
Γ_1/Γ_3	0.23083388	0.2332	1%
Γ_1/Γ_4	0.08578756	0.0908	6%
Γ_1/Γ_5	0.08578756	0.0850	1%
Γ_1/Γ_6	0.06597117	0.0678	3%
Γ_1/Γ_7	0.04007602	0.0419	5%
Γ_1/Γ_8	0.04007602	0.0419	5%
Γ_1/Γ_9	0.02820056	0.0294	4%
Γ_1/Γ_{10}	0.02820056	0.0294	4%

Comparison of known and computed ratios of clamped plate eigenvalues for squares (A. Weinstein)

	Theoretical low bound	Theoretical upper bound	Computed values	Error from upper bound
Γ_1/Γ_2	0.23482612	0.24229181	0.2448	1%
Γ_1/Γ_3	0.23482612	0.24229181	0.2439	1%
Γ_1/Γ_4	0.10704902	0.11152849	0.1148	3%
Γ_1/Γ_5	0.07292193	0.07533179	0.0789	5%
Γ_1/Γ_6	0.07198916	0.07506562	0.0769	2%
Γ_1/Γ_7	0.04404576	0.04824526	0.0513	6%
Γ_1/Γ_8	0.04404576	0.04824526	0.0510	6%
Γ_1/Γ_9	0.02784486	0.02948061	0.0327	11%
Γ_1/Γ_{10}	0.02784486	0.02948061	0.0320	9%

Other Methods of Computation

- Weinstein Method (late 30s)
- Weinstein-Aronszajn Method (mid 40s)
- Fichera Method of Orthogonal Invariants (60s, 70s)
- Bazley-Fox-Stadter (1967)
- J. McLaurin (1968)
- Kuttler Method (1972) (à la Weinberger)
- Bauer-Reis (1972)
- C. Wieners (1996)
- Aimi & Diligenti (1992) ("Buckling" à la Fichera)
- Weinberger (à la Weinberger (for Clamped) and à la Fichera (for Neumann))

Feature Vectors

λ represents any of the eigenvalues μ, Γ, Λ.

$$F_1(\Omega) = \left(\frac{\lambda_1}{\lambda_2}, \frac{\lambda_1}{\lambda_3}, \frac{\lambda_1}{\lambda_4}, \ldots, \frac{\lambda_1}{\lambda_n}\right)$$

$$F_2(\Omega) = \left(\frac{\lambda_1}{\lambda_2}, \frac{\lambda_2}{\lambda_3}, \frac{\lambda_3}{\lambda_4}, \ldots, \frac{\lambda_{n-1}}{\lambda_n}\right)$$

$$F_3(\Omega) = \left(\frac{\lambda_1}{\lambda_2} - \frac{d_1}{d_2}, \frac{\lambda_1}{\lambda_3} - \frac{d_1}{d_3}, \frac{\lambda_1}{\lambda_4} - \frac{d_1}{d_4}, \ldots, \frac{\lambda_1}{\lambda_n} - \frac{d_1}{d_n}\right)$$

Here $d_1 \leq d_2, \ldots \leq d_n$ are the first n e-values of a disk.

$$F_4(\Omega) = \left(\frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{2\lambda_1}, \frac{\lambda_4}{3\lambda_1}, \ldots, \frac{\lambda_{n+1}}{n\lambda_1}\right)$$

(F_4 scales down the Weyl growth of the eigenvalues.)

For clamped plate:

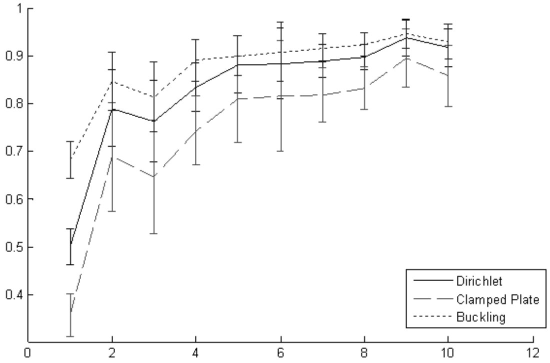
$$F_4(\Omega) = \left(\frac{\Gamma_2}{\Gamma_1}, \frac{\Gamma_3}{4\Gamma_1}, \frac{\Gamma_4}{9\Gamma_1}, \ldots, \frac{\Gamma_{n+1}}{n^2\Gamma_1}\right)$$

(F_4 scales down the Weyl growth of the eigenvalues.)

Experiments: Correct classification rates for hand-drawn shapes

	Dirichlet		Neumann		Stekloff	
n	F_1	F_2	F_1	F_2	F_1	F_2
4	60.0%	91.0%	87.5%	91.0%	40.5%	34.0%
8	94.0%	94.0%	94.0%	94.0%	45.0%	41.5%
12	94.5%	93.5%	94.5%	94.0%	50.0%	42.0%
16	92.5%	78.5%	92.5%	91.0%	61.0%	57.5%
20	95.5%	94.5%	95.5%	94.5%	55.5%	56.0%

Experiments: Standard Deviation of the first F_2 features for 100 triangles using Dirichlet, Clamped and Buckling Eigenvalues



Experiments

- 40 disks, triangles, rectangles, ellipses, diamonds, and squares (total 240 images) of different sizes and orientations hand written and scanned into computer: Noisy and irregular boundaries
- 300 additional computer generated images of the same shapes were added to the database (aspect ratios vary from 2 to 2.5 for elongated figures): Noise free
- These 300 computer generated images were used to train the neural network with
- Dirichlet, Neumann, Clamped, and Buckling eigenvalues were computed and $n = 20$ $F_1, \ldots, F_4$ feature vectors from each of the six classes were generated.
- A simple neural network was trained with the 300 computer generated images
- Another 300 computer generated images and the 240 hand-written ones were used in the validation phase

Results for Computer Generated and Hand-Drawn Shapes

Dirichlet Features

	F1		F2		F3		F4	
N	CG	HD	CG	HD	CG	HD	CG	HD
4	98.0	95.1	97.3	89.2	96.0	91.7	97.3	95.8
8	99.3	94.1	99.3	92.4	98.0	94.4	99.3	96.2
12	99.0	94.4	99.3	92.4	98.7	93.8	99.3	96.5
16	99.3	95.8	99.3	94.1	99.0	96.2	99.7	96.5
20	99.7	96.5	99.3	87.8	99.3	96.5	99.0	97.2
Avrg	**99.1**	**95.2**	**98.9**	**91.2**	**98.2**	**94.5**	**98.9**	**96.4**

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Clamped Plate Features

	F1		F2		F3		F4	
N	CG	HD	CG	HD	CG	HD	CG	HD
4	93.3	90.3	94.0	76.4	93.7	89.6	95.7	92.7
8	95.7	87.5	96.0	91.7	95.3	90.3	96.3	91.7
12	97.7	89.6	97.0	92.0	96.0	89.6	96.0	93.8
16	98.3	92.4	98.3	80.9	98.0	89.6	98.7	92.4
20	98.0	90.3	98.7	84.0	99.0	91.3	97.3	95.1
Avrg	**96.6**	**90.0**	**96.8**	**85.0**	**96.4**	**90.1**	**96.8**	**93.1**

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Neumann Features

	F1		F2		F3		F4	
N	CG	HD	CG	HD	CG	HD	CG	HD
4	99.0	93.1	99.0	95.1	98.3	93.1	99.0	96.5
8	99.3	97.2	99.7	97.2	99.3	98.3	99.3	97.9
12	99.7	97.9	99.0	96.5	100	98.3	99.7	98.3
16	100	98.6	98.3	96.5	100	97.6	99.7	98.6
20	99.7	97.9	98.7	95.8	100	96.5	99.7	99.0
Avrg	**99.6**	**96.9**	**98.9**	**96.2**	**99.5**	**96.8**	**99.5**	**98.1**

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Buckling Plate Features

	F1		F2		F3		F4	
N	CG	HD	CG	HD	CG	HD	CG	HD
4	98.0	91.3	92.7	86.8	92.7	87.8	93.7	92.7
8	93.7	91.7	94.7	89.9	93.7	87.2	93.3	90.3
12	94.7	90.3	94.0	85.8	96.0	91.3	95.0	95.8
16	94.3	92.4	95.3	89.6	96.7	92.4	96.0	95.1
20	97.7	94.4	94.7	85.4	95.7	94.1	96.0	94.1
Avrg	**95.7**	**92.0**	**94.3**	**87.5**	**95.0**	**90.6**	**94.8**	**93.6**

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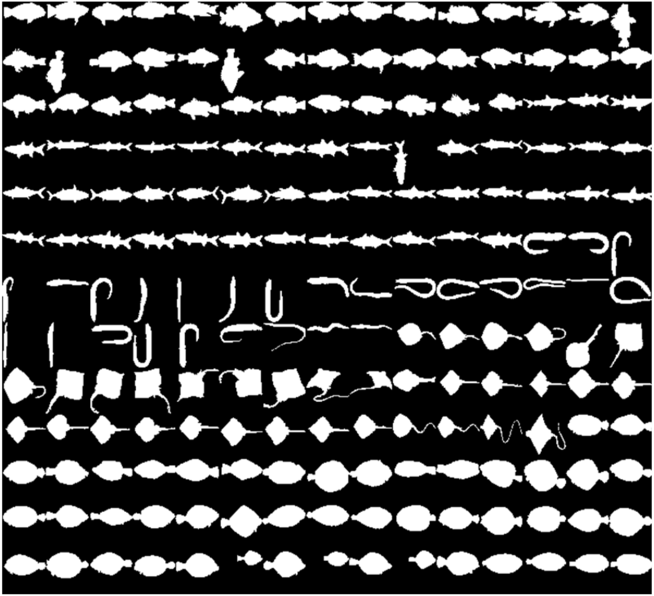
Shape Queries Using Image Databases (SQUID)

http://www.ee.surrey.ac.uk/CVSSP/demos/css/demo.html



Experiments on SQUID Database

- Dirichlet, Neumann, buckling plate, and clamped plate features were generated for 195 images of sting ray, snapper, eel, mullet, and flounder-like fish.
- A series of simple neural networks were trained on 65 images from this dataset and tested on the remaining 130 images.
- $n = 4, 8, 12, 16$, and 20 eigenvalues were used as inputs into the neural net for each of the model problems.

Experiments on SQUID Database: Correct Classification Rate of the Fish

	Neumann features				Dirichlet features				Buckling features				Clamped plate features			
N	F1	F2	F3	F4	F1	F2	F3	F4	F1	F2	F3	F4	F1	F2	F3	F4
4	88	85	89	87	92	95	94	94	87	84	93	85	90	87	92	89
8	94	89	91	91	93	94	94	94	89	83	92	90	90	86	91	90
12	91	86	90	92	93	93	94	97	88	84	93	90	92	80	93	96
16	95	87	95	96	94	91	92	98	91	85	92	92	92	85	93	94
20	95	88	95	97	94	88	93	98	92	87	93	93	92	81	92	94

Unified Approach to Universal Inequalities

- H. C. Yang inequality is just a discriminant condition in an abstract (purely) algebraic scheme.
- Universal inequalities for Dirichlet eigenvalues of Yang-type and versions recently proved for the clamped plate problem (proved by Wang-Xia, Wu-Cao, etc.) are corollaries to this setting.
- This work generalizes earlier joint work with M. Ashbaugh (Pac. J. Math., 2004)

Setting:

- $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\ ,\ \rangle$,

- $A : \mathcal{D} \subset \mathcal{H} \to \mathcal{H}$ a self-adjoint operator defined on a dense domain $\mathcal{D}$ which is semibounded below and has a discrete spectrum $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$.

- $\{T_k : \mathcal{D} \to \mathcal{H}\}_{k=1}^N$: a collection of skew-symmetric operators,
- $\{B_k : T_k(\mathcal{D}) \to \mathcal{H}\}_{k=1}^N$ a collection of symmetric operators which leave $\mathcal{D}$ invariant, and $\{u_i\}_{i=1}^\infty$ the normalized eigenvectors of A, u_i corresponding to λ_i. We may further assume that $\{u_i\}_{i=1}^\infty$ is an orthonormal basis for $\mathcal{H}$.
- $[A, B]$ denotes the commutator of two operators defined by $[A, B] = AB - BA$, and $\|u\| = \sqrt{\langle u, u\rangle}$.

Main Theorem

Define:

$$\beta_i = \sum_{k=1}^{N} \langle [B_k, T_k] u_i, u_i \rangle,$$

$$\rho_i = \sum_{k=1}^{N} \langle [A, B_k] u_i, B_k u_i \rangle,$$

and

$$\Lambda_i = \sum_{k=1}^{N} \| T_k u_i \|^2.$$

Statement:

The eigenvalues $\{\lambda_i\}$ of the operator A satisfy the following inequality

$$\left(\sum_{i=1}^{m} \beta_i \left(\lambda_{m+1} - \lambda_i \right)^2 \right)^2 \leq 4 \left(\sum_{i=1}^{m} \rho_i \left(\lambda_{m+1} - \lambda_i \right)^2 \right) \left(\sum_{i=1}^{m} \Lambda_i \left(\lambda_{m+1} - \lambda_i \right) \right)$$

Consequences:

Facts:

- $$\rho_i = \frac{1}{2} \sum_{k=1}^{N} \langle [B_k, [A, B_k]] u_i, u_i \rangle.$$
- When $T_k = [A, B_k]$, one has $\beta_i = 2\rho_i$, $\Lambda_i = \sum_{k=1}^{N} \|[A, B_k] u_i\|^2$. In this case, the statement of the theorem reduces to the familiar H. C. Yang inequality in the abstract setting (Levitin-Parnovski, 2001, Ashbaugh-H., 2004, Harrell-Stubbe 2009):

$$\sum_{i=1}^{m} \rho_i \left(\lambda_{m+1} - \lambda_i\right)^2 \leq \sum_{i=1}^{m} \Lambda_i \left(\lambda_{m+1} - \lambda_i\right).$$

Flavor of the Proof

- Start with Rayleigh-Ritz for λ_{m+1}

$$\lambda_{m+1} \leq \frac{\langle A\phi, \phi \rangle}{\langle \phi, \phi \rangle}$$

-

$$\phi_i = Bu_i - \sum_{j=1}^{m} a_{ij} u_j, \tag{7}$$

where $a_{ij} = \langle Bu_i, u_j \rangle$

- $a_{ji} = \overline{a_{ij}}$.
- Let $b_{ij} = \langle [A, B] u_i, u_j \rangle$, then

$$b_{ij} = -\overline{b_{ji}} = (\lambda_j - \lambda_i)\, a_{ij}.$$

Flavor of the Proof, cont'd

- R-R reduces to:

$$\lambda_{m+1} - \lambda_i \le \frac{\langle [A,B]u_i, \phi \rangle}{\langle \phi, \phi \rangle}.$$

- Also

$$\langle [A,B]u_i, \phi_i \rangle = \langle [A,B]u_i, Bu_i \rangle - \sum_{j=1}^{m} (\lambda_j - \lambda_i) \, |a_{ij}|^2.$$

- Since T is an antisymmetric operator

$$Re\langle \phi_i, Tu_i \rangle = Re\langle \phi_i, Tu_i - \sum_{j=1}^{m} t_{ij} u_j \rangle,$$

for $t_{ij} = \langle Tu_i, u_j \rangle$ (since $\langle \phi_i, u_j \rangle = 0$, for $j = 1, 2, \ldots, m$.)

Flavor of the Proof, cont'd

- For $\gamma > 0$:

$$Re\langle \phi_i, Tu_i \rangle \leq \frac{1}{2\gamma} \left(\lambda_{m+1} - \lambda_i\right) \|\phi_i\|^2 + \frac{\gamma}{2\left(\lambda_{m+1} - \lambda_i\right)} \left(\| Tu_i \|^2 - \sum_{j=1}^{m} |t_{ij}|^2 \right)$$

-

$$\begin{aligned} \left(\lambda_{m+1} - \lambda_i\right)^2 Re\langle \phi_i, Tu_i \rangle &\leq \frac{1}{2\gamma} \left(\lambda_{m+1} - \lambda_i\right)^3 \|\phi_i\|^2 \\ &+ \frac{\gamma}{2} \left(\lambda_{m+1} - \lambda_i\right) \left(\| Tu_i \|^2 - \sum_{j=1}^{m} |t_{ij}|^2 \right). \end{aligned}$$

Flavor of the Proof, cont'd

- Put things together to get

$$
\begin{aligned}
& (\lambda_{m+1} - \lambda_i)^2 \, Re \, \langle \phi_i, Tu_i \rangle \\
\leq \quad & \frac{1}{2\gamma} \, (\lambda_{m+1} - \lambda_i)^2 \left(\langle [A, B] u_i, Bu_i \rangle - \sum_{j=1}^{m} (\lambda_j - \lambda_i) \, |a_{ij}|^2 \right) \\
+ \quad & \frac{\gamma}{2} \, (\lambda_{m+1} - \lambda_i) \left(\| Tu_i \|^2 - \sum_{j=1}^{m} |t_{ij}|^2 \right).
\end{aligned}
$$

- .. after a series of steps, one is led to:

$$
\begin{aligned}
\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^2 \, \langle [B, T] u_i, u_i \rangle \quad \leq \quad & \frac{1}{\gamma} \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^2 \, \langle [A, B] u_i, Bu_i \rangle \\
+ \quad & \gamma \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i) \, \| Tu_i \|^2.
\end{aligned}
$$

- Restore the dependence of T and B on the index $k = 1, \ldots N$, then sum on k

Flavor of the Proof, cont'd

- We are led to:

$$\sum_{i=1}^{m} \beta_i \left(\lambda_{m+1} - \lambda_i\right)^2 \leq \frac{1}{\gamma} \sum_{i=1}^{m} \rho_i \left(\lambda_{m+1} - \lambda_i\right)^2 + \gamma \sum_{i=1}^{m} \Lambda_i \left(\lambda_{m+1} - \lambda_i\right)$$

- Reduce to a quadratic statement in γ which is always ≥ 0, so the discriminant ≤ 0. This is the statement of the theorem.

Thank you!